Iracionalne funkcije

(*) Integrali oblika $\int R\left(x, \sqrt{ax^2+bx+c}\right) dx$ se rješavaju na 2 načina: Ojlerove / Trigonometrijske smjene

Ojlerove smjene

a) $a>0 \Rightarrow$ smjena $\sqrt{ax^2+bx+c} = \pm x\sqrt{a} \pm t \;/^2$

$$x = \frac{t^2-c}{b-2\sqrt{a}t} \Rightarrow dx = (\;)' dt$$

b) $c>0 \Rightarrow$ smjena $\sqrt{ax^2+bx+c} = xt + \sqrt{c} \;/^2$

$$x = \frac{2\sqrt{c}\,t-b}{a-t^2} \Rightarrow dx = (\;)' dt$$

c) ako su x_1 i x_2 nule polinoma ax^2+bx+c;

tj. ako je $ax^2+bx+c = a(x-x_1)(x-x_2)$

$\Rightarrow$ smjena $\sqrt{ax^2+bx+c} = t\,(x-x_0)$ ili x_1 ili x_2

Trigonometrijske smjene

$\int R\left(x, \sqrt{ax^2+bx+c}\right)dx$ se svodi na 1. od sl. 3 integrala

$$ax^2+bx+c = a\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}+\frac{c}{a}-\frac{b^2}{4a^2}\right) =$$

$$= a\left(x+\frac{b}{2a}\right)^2 + \left(c-\frac{b^2}{4a}\right) \rightarrow \frac{4ac-b^2}{4a}$$

smjena $x+\frac{b}{2a} = t$

a) $\int R^*\left(t, \sqrt{1-t^2}\right)dt$ b) $\int R^*\left(t, \sqrt{t^2-1}\right)dt$ ili c) $\int R^*\left(t, \sqrt{1+t^2}\right)$

smjena $t = \sin u$ smjena $t = \frac{1}{\sin u}$ smjena $t = \operatorname{tg} u$

(*) Integracija diferencijalnog binoma $\int x^m (a+bx^n)^p dx$

a) $p \in \mathbb{Z} \Rightarrow$ Njutnov obrazac

b) $\frac{m+1}{n} \in \mathbb{Z} \Rightarrow$ smjena $(a+bx^n) = t^s$; $p = \frac{r}{s}$ $\rightarrow$ oslobodi li se korijena $(t^s)^{\frac{r}{s}} = t^r$

c) $\frac{m+1}{n} + p \in \mathbb{Z} \Rightarrow$ smjena:

$\frac{a}{x^n} + b = t^s$, $p = \frac{r}{s}$

I $\int R(x, \sqrt{ax^2+bx+c})\,dx$

1) $a>0 \Rightarrow \sqrt{ax^2+bx+c} = \sqrt{a}\,x + t$

2) $c>0 \Rightarrow \sqrt{ax^2+bx+c} = xt + \sqrt{c}$

3) ako je α nula ~~poli~~ trinoma ax^2+bx+c

$\sqrt{ax^2+bx+c} = t\cdot(x-\alpha)$

more $\sqrt{ax^2+bx+c}$ da se ne uklapa u ove smjene ali nema smisla

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46.

$$\int \frac{dx}{\sqrt{1+x^2}}$$

ovdje lakša 1. Ojlerova smjena

2. Ojlerova smjena

$\sqrt{1+x^2} = xt+1 \quad /^2$

$\cancel{1}+x^2 = x^2t^2 + 2xt + \cancel{1}$

$x = xt^2 + 2t$

$x(1-t^2) = 2t$

$x = \dfrac{2t}{1-t^2}$

$$dx = \frac{2(1-t^2) - 2t(-2t)}{(1-t^2)^2}\,dt = \frac{2-2t^2+4t^2}{(1-t^2)^2}\,dt = 2\cdot\frac{1+t^2}{(1-t^2)^2}\,dt$$

$$\sqrt{1+x^2} = xt+1 = \frac{2t}{1-t^2}\cdot t+1 = \frac{2t^2+1-t^2}{1-t^2} = \frac{1+t^2}{1-t^2}$$

$$\sqrt{1+x^2} = xt+1 \Rightarrow t = \frac{\sqrt{1+x^2}-1}{x}$$

$$\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{2\cdot\frac{\cancel{1+t^2}}{(1-t^2)^{\cancel{2}}}\,dt}{\frac{\cancel{1+t^2}}{\cancel{1-t^2}}} = 2\int \frac{dt}{1-t^2} = 2\int \frac{dt}{(1-t)(1+t)}$$

$$\frac{A}{1-t} + \frac{B}{1+t} = \frac{1}{(1-t)(1+t)}$$

$1 = A(1+t) + B(1-t)$

$A-B=0 \Rightarrow A=B$

$A+B=1$

$A=\frac{1}{2} \quad B=\frac{1}{2}$

$$2\int \frac{dt}{(1-t)(1+t)} = 2\cdot\frac{1}{2}\int\frac{dt}{1-t} + \frac{1}{2}\int\frac{dt}{1+t} =$$

$$= -\ln|1-t| + \ln|1+t| + C =$$

$$= \ln\left|\frac{1+\frac{\sqrt{1+x^2}-1}{x}}{1-\frac{\sqrt{1+x^2}-1}{x}}\right| + C$$

možemo da sredimo a ne moramo

$$\frac{1+\frac{\sqrt{1+x^2}-1}{x}}{1-\frac{\sqrt{1+x^2}-1}{x}} = \frac{\frac{x+\sqrt{1+x^2}-1}{\cancel{x}}}{\frac{x-\sqrt{1+x^2}-1}{\cancel{x}}} = \frac{x+\sqrt{1+x^2}-1}{x-\sqrt{1+x^2}-1} \cdot \frac{\sqrt{1+x^2}+x+1}{\sqrt{1+x^2}+x+1}$$

Racionališemo

$$= \frac{(\sqrt{x^2+1}+x)^2-1}{(x+1)^2-(1+x^2)} = \frac{\cancel{1}+x^2+2x\sqrt{1+x^2}+\cancel{x^2}\cancel{-1}}{\cancel{x^2}+2x\cancel{+1}\cancel{-1}\cancel{-x^2}} =$$

$$= \frac{2x^2+2x\sqrt{1+x^2}}{2x} = \frac{\cancel{2x}(x+\sqrt{1+x^2})}{\cancel{2x}} = x+\sqrt{1+x^2}$$

$$I = \ln|x+\sqrt{1+x^2}| + C$$

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(47) $\displaystyle\int \frac{dx}{x+\sqrt{1+x+x^2}}$

1. Ojlerova smjena

$$\sqrt{1+x+x^2} = -x+t \ /^2$$

$$t = x+\sqrt{1+x}$$

$$1+x+x^2 = x^2-2xt+t^2$$

$$x+2xt = t^2-1 \rightarrow x = \frac{t^2-1}{1+2t}$$

$$dx = \frac{2t(1+2t)-(t^2-1)\cdot 2}{(1+2t)^2}\,dt$$

$$dx = \frac{2t+4t^2-2t^2}{(1+2t)^2}\,dt = 2\frac{t^2+t+1}{(1+2t)^2}\,dt$$

$$(t^2+t+1):(1+2t) =$$

$$(12t^2+8t+1)$$

$$2\int\frac{\frac{t^2+t+1}{(1+2t)^2}}{t}\,dt = 2\int\frac{t^2+t+1}{t(1+2t)^2}\,dt =$$

$$(t^2+t+1):(2t+1) = \frac{1}{2}t+\frac{1}{4}$$

$$\frac{-t^2-\frac{1}{2}t}{\quad\frac{1}{2}t+1}$$

$$\frac{-\frac{1}{2}t-\frac{1}{4}}{\frac{3}{4}}$$

$$2t+1$$

$$= 2\int\frac{(2t+1)(\frac{1}{2}t+\frac{1}{4})+\frac{3}{4}}{t(1+2t)^2}\,dt$$

$$t^2+t+1$$

$$t(4t^2+4t+1)$$

$$4t^3+4t^2+1 \ /'$$

$$t$$

$$-t^2-1$$

$$(t^2+t+1):t = (t+1)+\frac{1}{2}$$

$$-t^2$$

$$12t^2+8t+1=0$$

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48. $I = \int \frac{x\,dx}{\left(\sqrt{7x-10-x^2}\right)^3} = \left[-(x^2 - 7x + 10) = \frac{7 \pm \sqrt{b^2-4ac}}{2a} = \frac{7 \pm 3}{2} = \begin{cases} 5 \\ 2 \end{cases} = -(x^2-7x+10) = -(x-2)(x-5) \right]$

3. Ojlerova smjena:

$\sqrt{-x^2+7x-10} = t(x-2) \;/^2$

$-x^2+7x-10 = t^2(x^2-4x+4)$

$-x^2+7x-10 = t^2x^2 - 4t^2x + 4t^2$

$-(x-\cancel{2})(x-5) = t^2(x-2)^{\cancel{2}}$

$-(x-5) = t^2(x-2)$

$-x+5 = t^2x - 2t^2$

$x = \frac{5+2t^2}{1+t^2} \rightarrow dx = \frac{4t(1+t^2)-2t(5+2t^2)}{(1+t^2)^2}\,dt$

$dx = \frac{4t-10t}{(1+t^2)^2}\,dt = \frac{-6t\,dt}{(1+t^2)^2}$

$\sqrt{-x^2+7x-10} = t(x-2) = t \cdot \left(\frac{5+2t^2}{1+t^2} - 2\right) = t \cdot \frac{3}{1+t^2}$

$I = \int \frac{\frac{5+2t^2}{1+\cancel{t^2}} \cdot \frac{-6t}{(1+t^2)^2}}{\frac{27t^3}{(1+t^2)^3}}\,dt = -\frac{2}{9}\int \frac{5+2t^2}{t^2}\,dt =$

$= -\frac{2}{9}\cdot 5\int \frac{dt}{t^2} - \frac{4}{9}\int dt = +\frac{2}{9}\cdot 5 \cdot \frac{1}{t} - \frac{4}{9}t + C =$

$= \frac{10}{9}\cdot\frac{(x-2)}{\sqrt{-x^2+7x-10}} - \frac{4}{9}\cdot\frac{\sqrt{-x^2+7x-10}}{x-2} + C$

II $\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}\,dx$

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49) $\int \frac{x^2}{\sqrt{x^2+x+1}}\,dx$

$$\boxed{\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}\,dx = Q_{(n-1)}(x)\cdot\sqrt{ax^2+bx+c} + k\int\frac{dx}{\sqrt{ax^2+bx+c}}}$$

$$\int \frac{x^2+x+1-x-1}{\sqrt{x^2+x+1}}\,dx = \int \sqrt{x^2+x+1}\,dx - \int \frac{x+1}{\sqrt{x^2+x+1}}\,dx$$

$x^2+x+1 = 2x$

$$=\frac{2}{3}\left(\sqrt{x^2+x+1}\right)^3 - \int \frac{\frac{1}{2}(2x+1)+\frac{1}{2}}{\sqrt{x^2+x+1}}\,dx = \quad \frac{3}{2}$$

$$=\frac{2}{3}\left(\sqrt{x^2+x+1}\right)^3 - \frac{1}{2}\int\frac{2x+1}{\sqrt{x^2+x+1}}\,dx - \frac{1}{2}\int\frac{dx}{\sqrt{x^2+x+1}} =$$

$$=\frac{2}{3}\left(\sqrt{x^2+x+1}\right)^3 + \frac{1}{2}\cdot 2\sqrt{x^2+x+1} - \frac{1}{2}\int\frac{dx}{\sqrt{x^2+x+1}} \quad \sqrt{x^2+1}$$

Ojlerove smjene; I smjena, $a>0$

$\sqrt{x^2+x+1} = x+t \;/^2$

$\cancel{x^2}+x+1 = \cancel{x^2}+2xt+t^2$

$x+1 = 2xt+t^2$

$x-2xt = t^2-1 \Rightarrow x(1-2t) = t^2-1$

$$x = \frac{t^2-1}{1-2t}$$

$$dx = \frac{2t(1-2t)+2(t^2-1)}{(1-2t)^2}\,dt = \frac{-4t^2+2t+2t^2-1}{(1-2t)^2}\,dt =$$

$$= \frac{-2t^2+2t-1}{(1-2t)^2}\,dt = \left(\frac{-2t^2}{(1-2t)^2} - \frac{1-2t}{(1-2t)^2}\right)dt$$

(49) $$\int \frac{x^2}{\sqrt{x^2+x+1}}\,dx = (Ax+B)\sqrt{x^2+x+1} + k\int \frac{dx}{\sqrt{x^2+x+1}}$$

$$\frac{x^2}{\sqrt{x^2+x+1}} = A\sqrt{x^2+x+1} + (Ax+B)\cdot\frac{2x+1}{2\sqrt{x^2+x+1}} + \frac{k}{\sqrt{x^2+x+1}} \Bigg/ \cdot \frac{2}{\sqrt{x^2+x+1}}$$

$$2x^2 = 2A(x^2+x+1) + (Ax+B)(2x+1) + 2k$$

$$2x^2 = x^2(2A+2A) + (2A+2B+A) + 2A+B+2k$$

$$4A = 2 \Rightarrow A = \frac{1}{2}$$

$$3A + 2B = 0 \Rightarrow B = -\frac{3}{4}$$

$$2A+B+2k=0 \Rightarrow k = -\frac{1}{8}$$

$$I = \left(\frac{1}{2}x - \frac{3}{4}\right)\cdot\sqrt{x^2+x+1} - \frac{1}{8}\int\frac{dx}{\sqrt{x^2+x+1}}$$

$$-\frac{1}{8}\int\frac{dx}{\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}} = -\frac{1}{8}\int\frac{dx}{\sqrt{\frac{3}{4}\left(\left(\frac{2}{\sqrt3}x+\frac{1}{\sqrt3}\right)^2+1\right)}} =$$

$$= -\frac{1}{4}\cdot\frac{1}{\sqrt3}\int\frac{dx}{\sqrt{\left(\frac{2}{\sqrt3}x+\frac{1}{\sqrt3}\right)^2+1}} = \left[\begin{array}{l}\frac{2}{\sqrt3}x+\frac{1}{\sqrt3} = t\\ dx = \frac{\sqrt3}{2}dt\end{array}\right] =$$

$$= -\frac{1}{4\cancel{\sqrt3}}\cdot\frac{\cancel{\sqrt3}}{2}\int\frac{dt}{\sqrt{t^2+1}} = -\frac{1}{8}\int\frac{dt}{\sqrt{t^2+1}} =$$

$$= -\frac{1}{8}\cdot \ln\left|x+\sqrt{x^2+1}\right| + C$$

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III $\displaystyle\int \frac{Mx+N}{(x-\alpha)^n\sqrt{ax^2+bx+c}}\,dx \rightarrow$ smjena $\boxed{x-\alpha=\frac{1}{t}}$

(50)

$$I=\int \frac{x}{(x-1)^2\cdot\sqrt{-x^2+2x+1}}\,dx=\left[\begin{array}{l} x-1=\frac{1}{t}\\ x=1+\frac{1}{t}\\ dx=\frac{-dt}{t^2}\end{array}\right]=$$

$$=\left[-x^2+2x+1=\frac{-(t+1)^2}{t^2}+\frac{2t+2}{t}+1=\right.$$

$$\left.=\frac{-t^2-2t-1+2t^2+2t+t^2}{t^2}=\frac{2t^2-1}{t^2}\right]=$$

$$\rightarrow I=-\int\frac{\frac{t+1}{\cancel{t}}}{\frac{1}{\cancel{t^2}}\sqrt{\frac{2t^2-1}{\cancel{t^2}}}}\cdot\frac{dt}{\cancel{t^2}}=-\int\frac{t+1}{\sqrt{2t^2-1}}\,dt=$$

$$=-\int\frac{\frac{1}{4}\,4t-1}{\sqrt{2t^2-1}}\,dt=-\frac{1}{4}\int\frac{4t-1}{\sqrt{2t^2-1}}\,dt+\int\frac{dt}{\sqrt{2t^2-1}}$$

$$=-\frac{1}{4}\int\frac{4t}{\sqrt{2t^2-1}}\cdot dt+\int\frac{dt}{\sqrt{2t^2-1}}=$$

$2t^2-1=z$ $\qquad\qquad$ $\sqrt{2}t=s$

$4t\,dt=dz$ $\qquad\qquad$ $dt=\frac{1}{\sqrt{2}}ds$

$$=-\frac{1}{4}\int\frac{dz}{\sqrt{z}}+\frac{1}{\sqrt{2}}\int\frac{ds}{\sqrt{s^2-1}}=$$

$$=+\frac{1}{2}\cdot\sqrt{z}+\frac{1}{\sqrt{2}}\cdot\ln\left|s+\sqrt{s^2-1}\right|+C=$$

$$=\frac{1}{2}\sqrt{2t^2-1}+\frac{1}{\sqrt{2}}\ln\left|\sqrt{2}t+\sqrt{2t^2-1}\right|+C=$$

$$=\frac{1}{2}\sqrt{2(x-1)^2-1}+\frac{1}{\sqrt{2}}\ln\left|\sqrt{2}(x-1)+\sqrt{2(x-1)^2-1}\right|+C$$

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(51) $$\int \frac{x^2+\sqrt{1+x}}{\sqrt[3]{1+x}}\,dx = \left[\begin{array}{l} 1+x = t^6 \\ dx = 6t^5\,dt \end{array}\right] =$$

$$= 6\int \frac{(t^6-1)^2+t^3}{t^2}\cdot t^5\,dt =$$

$$= 6\int \frac{(t^6-1)^2+t^3}{1}\cdot t^3\,dt = 6\int\left((t^6-1)^2+t^3\right)\cdot t^3\,dt$$

zapamtiti ideju!!!

(52.) $$\int \sqrt{\frac{x+1}{x-1}}\,dx = \left[\begin{array}{l} \frac{x+1}{x-1} = t^2 \qquad x - t^2x = -1-t^2 \\ x+1 = t^2x - t^2 \qquad x = \frac{-1-t^2}{1-t^2} = \frac{t^2+1}{t^2-1} \\ dx = \frac{2t(t^2-1)-2t(t^2+1)}{(t^2-1)^2}\,dt \\ dx = \frac{-4t}{(t^2-1)^2}\,dt \end{array}\right]$$

$$= -4\int t\cdot\frac{t}{(t^2-1)^2}\,dt =$$

rac → 4 sabirka

lakše parcijalna integracija

$$\mathrm{I} = -4\int t\cdot\frac{t}{(t^2-1)^2}\,dt = -4\int t\cdot\frac{(t+1)-1}{(t-1)^2(t+1)^2}\,dt =$$

$$= -4\int\frac{t\cdot(t+1-1)}{(t-1)^2(t+1)^2}\,dt = -4\int\frac{t^2+t-t}{(t-1)^2(t+1)^2}\,dt =$$

$$= -4\int\frac{t^2+t}{(1-t^2)(t+1)^2} -$$

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$$-4\int t\cdot\frac{t}{(t^2-1)^2}\,dt = \left[\begin{array}{l} u=t \Rightarrow du=dt \\ V=\int \frac{t}{(t^2-1)^2}\,dt = \left[\begin{array}{l} t^2-1=z \\ t\,dt=\frac{1}{2}dz\end{array}\right] = \\ = \frac{1}{2}\int\frac{dz}{z^2} = -\frac{1}{2}\cdot\frac{1}{z} = -\frac{1}{2}\cdot\frac{1}{t^2-1}\end{array}\right]$$

$$= -4\left(-\frac{1}{2}\cdot\frac{t}{t^2-1}+\frac{1}{2}\int\frac{dt}{t^2-1}\right) = \left[\begin{array}{l}\frac{x+1}{x-1}=t^2 \\ \sqrt{\frac{x+1}{x-1}}=t\end{array}\right] =$$

$$= 2\cdot\frac{t}{t^2-1} - 2\,\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C =$$

$$= 2\cdot\frac{\sqrt{\frac{x+1}{x-1}}}{\frac{x+1}{x-1}-1} - 2\cdot\frac{1}{2}\ln\left|\frac{\sqrt{\frac{x+1}{x-1}}-1}{\sqrt{\frac{x+1}{x-1}}+1}\right| + C =$$

$$= 2\sqrt{\frac{x+1}{x-1}}\cdot\frac{x-1}{2} - \ln\left|\frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}\right| + C$$

IV $\int\frac{Ax+B}{\sqrt{ax^2+bx+c}}\,dx,\quad \int\frac{dx}{\sqrt{ax^2+bx+c}}$

53.

$$\int\frac{x+3}{\sqrt{4x^2+4x+3}}\,dx = \int\frac{\frac{1}{8}(8x+4)+3-\frac{1}{2}}{\sqrt{4x^2+4x+3}}\,dx =$$

$$= \frac{1}{8}\int\frac{8x+4}{\sqrt{4x^2+4x+3}}\,dx + \frac{5}{2}\int\frac{dx}{\sqrt{4x^2+4x+3}} =$$

$$\left[\begin{array}{ll} 4x^2+4x+3=t & 4x^2+4x+3 = 4\left(x^2+x+\frac{3}{4}\right) = \\ (8x+4)\,dx = dt & = 4\left(x^2+x+\frac{1}{4}+\frac{3}{4}-\frac{1}{4}\right) = \\ & = 4\left(\left(x+\frac{1}{2}\right)^2+\frac{1}{2}\right)\end{array}\right]$$

$$= \frac{1}{8}\int\frac{dt}{\sqrt{t}} + \frac{5}{2}\int\frac{dx}{\sqrt{4\left(\left(x+\frac{1}{2}\right)^2+\frac{1}{2}\right)}} =$$

$$= \frac{1}{8} \cdot 2\sqrt{t} + \frac{5}{2} \cdot \frac{1}{2} \int \frac{dx}{\sqrt{(x+\frac{1}{2})^2 + \frac{1}{2}}} = \begin{bmatrix} x+\frac{1}{2} = \frac{1}{\sqrt{2}} z \\ dx = \frac{1}{\sqrt{2}} dz \\ z = \frac{2x+1}{\sqrt{2}} \end{bmatrix} =$$

$$= \frac{1}{4}\sqrt{t} + \frac{5}{4\sqrt{2}} \cdot \sqrt{2} \cdot \int \frac{dz}{\sqrt{z^2+1}} =$$

$$= \frac{1}{4} \cdot \sqrt{4x^2+4x+3} + \frac{5}{4} \ln \left| z + \sqrt{z^2+1} \right| + C =$$

$$= \frac{1}{4} \sqrt{4x^2+4x+3} + \frac{5}{4} \ln \left| \frac{2x+1}{\sqrt{2}} + \sqrt{\frac{(2x+1)^2}{2} + 1} \right| + C$$

V

a) $\int R\left(x, x^{\frac{m_1}{n_1}}, \ldots, x^{\frac{m_s}{n_s}}\right) dx \rightarrow x = t^k,\ k = NZS(n_1, \ldots, n_s)$

b) $\int R\left(x, (ax+b)^{\frac{m_1}{n_1}}, \ldots, (ax+b)^{\frac{m_s}{n_s}}\right) dx \rightarrow ax+b = t^k$

c) $\int R\left(x, \left(\frac{ax+b}{cx+d}\right)^{\frac{m_1}{n_1}}, \ldots, \left(\frac{ax+b}{cx+d}\right)^{\frac{m_s}{n_s}}\right) dx \rightarrow \frac{ax+b}{cx+d} = t^k$

(54) $\int R\left(x, x^{\frac{m_1}{n_1}}, \ldots, x^{\frac{m_s}{n_s}}\right)$

predložena smjena: $x = t^k,\ k = NZS(n_1, n_2, \ldots, n_s)$

$$\int \frac{dx}{\sqrt{x} + \sqrt[3]{x}} = \begin{bmatrix} R(x, x^{\frac{1}{2}}, x^{\frac{1}{3}}) & x^{\frac{1}{2}} = t^3 \\ x = t^6 \rightarrow t = \sqrt[6]{x} & x^{\frac{1}{3}} = t^2 \\ dx = 6t^5\,dt & \end{bmatrix} =$$

$$= \int \frac{6t^5\,dt}{t^3 + t^2} = \int \frac{6t^{\cancel{5}\,3}\,dt}{\cancel{t^2}(1+t)} = 6\int \frac{t^3\,dt}{1+t} =$$

$$= 6\int \frac{t^3+1-1}{1+t}\,dt = 6\int \frac{\cancel{(t+1)}(t^2-t+1)}{\cancel{1+t}}\,dt - 6\int \frac{dt}{1+t} =$$

$$= 6\int (t^2 - t + 1)\,dt - 6\int \frac{dt}{1+t} =$$

$$= 2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\ln\left|\sqrt[6]{x} + 1\right| + C$$

(55.) $\int R\left(x, (ax+b)^{\frac{m_1}{n_1}}, \dots, (ax+b)^{\frac{m_s}{n_s}}\right) dx$

predložena smjena: $ax+b = t^k$, $k = NZS(n_1, \dots, n_s)$

$$\int \frac{x^2 + \sqrt{1+x}}{\sqrt[3]{1+x}}\,dx = \left[\begin{array}{ll} 1+x = t^6 & R\left(x, (1+x)^{\frac{1}{2}}, (1+x)^{\frac{1}{3}}\right) \\ dx = 6t^5\,dt & x = t^6 - 1 \\ (1+x)^{\frac{1}{2}} = t^3 & t = \sqrt[6]{1+x} = \\ (1+x)^{\frac{1}{3}} = t^2 & = (1+x)^{\frac{1}{6}} \end{array}\right]$$

$$= \int \frac{(t^6-1)^2 + t^3}{t^2} \cdot 6t^5\,dt = 6\int t^3 \cdot (t^{12} - 2t^6 + 1 + t^3)\,dt =$$

$$= 6 \cdot \int t^{15}\,dt - 12\int t^9\,dt + 6\int t^3\,dt + 6\int t^6\,dt =$$

$$= 6\,\frac{t^{16}}{16} - 12\,\frac{t^{10}}{10} + 6 \cdot \frac{t^4}{4} + 6 \cdot \frac{t^7}{7} + C =$$

$$= \frac{3}{8} \cdot \sqrt[3]{(1+x)^8} - \frac{6}{5} \cdot \sqrt[3]{(1+x)^5} + \frac{3}{2}\sqrt[3]{(1+x)^2} + \frac{6}{7}\sqrt[6]{(1+x)^7} + C$$

(56.) $\int R\left(x, \left(\frac{ax+b}{cx+d}\right)^{\frac{m_1}{n_1}}, \dots, \left(\frac{ax+b}{cx+d}\right)^{\frac{m_s}{n_s}}\right) dx$

predložena smjena: $\frac{ax+b}{cx+d} = t^k$, $k = NZS(n_1, n_2, \dots, n_s)$

$$\int \sqrt{\frac{x+1}{x-1}}\,dx = \left[\begin{array}{l} R\left(x, \left(\frac{x+1}{x-1}\right)^{\frac{1}{2}}\right) \longrightarrow \text{zadatak } 52 \\ \frac{x+1}{x-1} = t^2 \end{array}\right]$$

VI $\int x^m \cdot (a+bx^n)^p\,dx$, $a,b \in \mathbb{R}$, m,n,p → racionalni

1° $p \in \mathbb{Z}$; $p>0 \Rightarrow (a+bx^n)^p$ razvijamo po binomnoj formuli

$$(A+B)^n = \sum_{k=0}^{n} \binom{n}{k} A^n B^{n-k}$$

$p<0 \Rightarrow$ smjena $x = t^k$, $k = NZS$ (imenilaca brojeva n i m)

2° $\frac{m+1}{n} \in \mathbb{Z} \Rightarrow$ smjena $a+bx^n = t^k$, $k \to$ imenilac p

3° $\frac{m+1}{n} + p \in \mathbb{Z} \Rightarrow$ smjena $\frac{a}{x^n} + b = t^k$, k - imenilac p

57.

$$\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}\,dx = \int x^{-\frac{1}{2}} \cdot \left(1+x^{\frac{1}{4}}\right)^{\frac{1}{3}} dx =$$

$= \Big[$ $m = -\frac{1}{2}$; $n = \frac{1}{4}$; $p = \frac{1}{3}$

1° $p \notin \mathbb{Z}$ 2° $\frac{m+1}{n} = \frac{-\frac{1}{2}+\frac{2}{2}}{\frac{1}{4}} = \frac{\frac{1}{2}}{\frac{1}{4}} = \frac{4}{2} = 2 \in \mathbb{Z}$

$\Rightarrow$ smjena $a+bx^n = t^k$;

$1 + x^{\frac{1}{4}} = t^k \rightarrow 1 + x^{\frac{1}{4}} = t^3$

$x^{\frac{1}{4}} = t^3 - 1 \longrightarrow x = (t^3-1)^4$

$dx = 4(t^3-1)^3 \cdot 3t^2\,dt = 12t^2(t^3-1)^3\,dt$

$x^{-\frac{1}{2}} = (t^3-1)^{-2}$

$\left(1+x^{\frac{1}{4}}\right)^{\frac{1}{3}} = (t^3)^{\frac{1}{3}} = t$ $\Big]$

$$= \int (t^3-1)^{-2} \cdot t \cdot 12t^2 (t^3-1)^3\,dt =$$

$$= 12\int t^3(t^3-1)\,dt = 12\int t^6\,dt - 12\int t^3\,dt =$$

$$= 12\,\frac{t^7}{7} - 12\,\frac{t^4}{4} + C = \frac{12}{7}\cdot\sqrt[3]{\left(1+\sqrt[4]{x}\right)^7} - 3\cdot\sqrt[3]{\left(1+\sqrt[4]{x}\right)^4} + C$$

(58.) $$\int \frac{dx}{x^2\cdot\sqrt[3]{(2+x^3)^5}} = \int x^{-2}\cdot(2+x^3)^{-\frac{5}{3}}\,dx =$$

$= \quad m=-2;\quad n=3;\quad p=-\frac{5}{3}$

$1^\circ\ p \notin \mathbb{Z} \qquad 2^\circ\ \frac{m+1}{n} = \frac{-2+1}{3} = -\frac{1}{3} \notin \mathbb{Z}$

$3^\circ\ \frac{m+1}{n} + p = -\frac{1}{3} - \frac{5}{3} = -\frac{6}{3} = -2 \in \mathbb{Z}$

$\Rightarrow$ smjena $\frac{a}{x^n} + b = t^k;\quad k=3$

$\frac{2}{x^3} + 1 = t^3$! „Ključ" za integral

$\boxed{t = \sqrt[3]{\frac{2}{x^3}+1}} \qquad \frac{2}{x^3} = t^3 - 1$

$\frac{1}{x^3} = \frac{1}{2}\cdot(t^3-1) \rightarrow x^3 = \frac{2}{t^3-1} = 2\cdot(t^3-1)^{-1} \ /^{\frac{1}{3}}$

$\boxed{x = 2^{\frac{1}{3}}\cdot(t^3-1)^{-\frac{1}{3}}} \Rightarrow dx = 2^{\frac{1}{3}}\cdot\left(-\frac{1}{\cancel{3}}\right)\cdot(t^3-1)^{-\frac{4}{3}}\cdot \cancel{3}t^2 =$

$dx = -2^{\frac{1}{3}}\,t^2\,(t^3-1)^{-\frac{4}{3}}$

$2+x^3 = 2 + 2\cdot(t^3-1)^{-1} = 2\left(1+\frac{1}{t^3-1}\right)$

$(2+x^3)^{-\frac{5}{3}} = 2^{-\frac{5}{3}}\cdot t^{-5}\cdot(t^3-1)^{\frac{5}{3}}$

$$= -\int 2^{-\frac{2}{3}}\cdot(t^3-1)^{\frac{2}{3}}\cdot 2^{-\frac{5}{3}}\cdot t^{-5}\cdot(t^3-1)^{\frac{5}{3}}\cdot 2^{\frac{1}{3}}\cdot t^2\,(t^3-1)^{-\frac{4}{3}}\,dt =$$

$$= -\int 2^{-2}\cdot t^{-3}\cdot(t^3-1)\,dt = -\frac{1}{4}\int\frac{t^3-1}{t^3}\,dt =$$

$$= -\frac{1}{4}\sqrt[3]{\frac{2}{x^3}+1} - \frac{1}{8}\cdot\frac{1}{\sqrt[3]{\left(\frac{2}{x^3}+1\right)^2}} + C$$

VII a) $\int R(x, \sqrt{a^2-x^2})\,dx \rightarrow$ smjena $x = a\sin t$ ili $x = a\cos t$

b) $\int R(x, \sqrt{a^2+x^2})\,dx \rightarrow$ smjena $x = a\,tg\,t$

(59)

$$\int \sqrt{1-2x-x^2}\,dx = \begin{bmatrix} x^2-2x+1 = -(x^2+2x-1) = \\ = -(x^2+2x+1-1-1) = \\ = -((x+1)^2-2) = 2-(x+1)^2 \end{bmatrix} =$$

$$= \int \sqrt{2-(x+1)^2}\,dx = \begin{bmatrix} x+1=t \\ dx=dt \end{bmatrix} =$$

$$= \int \sqrt{2-t^2}\,dt = \begin{bmatrix} t=\sqrt{2}\sin z & dt=\sqrt{2}\cos z\,dz \\ \sin z = \frac{t}{\sqrt{2}} & \\ z = \arcsin\frac{t}{\sqrt{2}} & \end{bmatrix} =$$

$$= \int \sqrt{2-2\sin^2 z}\cdot\sqrt{2}\cdot\cos z\,dz = 2\int\sqrt{1-\sin^2 z}\cdot\cos z\,dz =$$

$$= 2\int\cos^2 z\,dz = 2\cdot\int\frac{1+\cos 2z}{2}\,dz = \cancel{\int dz + \int 2\int\cos 2z\,dz}$$

$$= \int dz + \int\cos 2z\,dz = z+\frac{1}{2}\sin 2z + C = z+\frac{1}{2}\,2\sin z\cos z + C =$$

$\rightarrow \cos x = \sqrt{1-\sin^2 x}$

$$= \arcsin\frac{t}{\sqrt{2}} + \frac{t}{\sqrt{2}}\cdot\sqrt{1-\frac{t^2}{2}} + C =$$

$$= \arcsin\frac{x+1}{\sqrt{2}} + \frac{x+1}{\sqrt{2}}\cdot\sqrt{1-\frac{(x+1)^2}{2}} + C$$

Trigonometrijske funkcije

* Kad ne znamo šta → I smjena ali obično može lakše

I $\int R(\sin x, \cos x)\, dx$

$$\operatorname{tg}\frac{x}{2} = t \;;\quad \sin x = \frac{2t}{1+t^2}$$

$$dx = \frac{2dt}{1+t^2} \quad \cos x = \frac{1-t^2}{1+t^2}$$

(60) $$\int \frac{dx}{(2+\cos x)\sin x} = \left[\begin{matrix} \operatorname{tg}\frac{x}{2} = t & \cos x = \frac{1-t^2}{1+t^2} \\ \sin x = \frac{2t}{1+t^2} & dx = \frac{2dt}{1+t^2} \end{matrix}\right]$$

$$= \int \frac{\frac{\cancel{2}dt}{\cancel{1+t^2}}}{\left(2+\frac{1-t^2}{1+t^2}\right)\cdot \frac{\cancel{2}t}{\cancel{1+t^2}}} = \int \frac{dt}{\frac{2(1+t^2)+1-t^2}{1+t^2}} = \int \frac{(1+t^2)\,dt}{t(3+t^2)} = I$$

$$\left[\frac{1+t^2}{t(3+t^2)} = \frac{A}{t} + \frac{Bx+C}{3+t^2} \right. \quad \left. \begin{matrix} A = \frac{1}{3} \\ B = \frac{2}{3} \\ C = 0 \end{matrix}\right]$$

$$I = \frac{1}{3}\int \frac{dt}{t} + \frac{2}{3}\int \frac{t\,dt}{3+t^2} = \left[\begin{matrix} t^2+3 = z \\ 2t\,dt = dz \end{matrix}\right] =$$

$$= \frac{1}{3}\ln|t| + \frac{\cancel{2}}{3}\cdot\frac{1}{\cancel{2}}\int \frac{dz}{z} = \frac{1}{3}\ln|t| + \frac{1}{3}\ln|z| + C =$$

$$= \frac{1}{3}\ln|t| + \frac{1}{3}\ln|t^2+3| + C = \frac{1}{3}\ln\left|\operatorname{tg}\frac{x}{2}\right| + \frac{1}{3}\ln\left|\operatorname{tg}^2\frac{x}{2}+3\right| + C$$

II $R(\sin x, -\cos x) = -R(\sin x, \cos x)$ $\boxed{\text{Smjena } \sin x = t}$

R → neparna po cos x → stepen cos neparan broj

(61) $$\int \frac{\cos^3 x}{2+\sin x}\,dx = \left[\begin{matrix} \sin x = t \\ \cos x\, dx = dt \end{matrix}\right] = \int \frac{\cos^2 x}{} \frac{1-t^2}{2+t}\,dt =$$

$$= -\int \frac{t^2-4+4-1}{2+t}\,dt = -\int \frac{(t-2)(t+2)}{t+2}\,dt - 3\int \frac{dt}{2+t}$$

$$= -\int (t-2)\,dt - 3\int \frac{dt}{t+2} = -\frac{t^2}{2} + 2t - 3\ln|t+2| + C =$$

$$= -\frac{1}{2}\sin^2 x + 2\sin x - 3\ln|\sin x + 2| + C$$

$\int V\,dU$

III $\int R(\sin x, \cos x) \rightarrow$ neparna po $\sin x$

smjena $\boxed{\cos x = t}$

IV $\int R(\sin x, \cos x)\,dx \rightarrow$ parna po sin i cos

smjena: $\operatorname{tg} x = t$ $\quad \sin x = \frac{t}{\sqrt{1+t^2}}$

$dx = \frac{dt}{1+t^2}$ $\quad \cos x = \frac{1}{\sqrt{1+t^2}}$

Koristi se kao gotova smjena

(62.) $\int \frac{\sin x - \cos x}{\sin x + 2\cos x}\,dx = \left[\begin{array}{l} R_1(\sin x, \cos x) = \frac{\sin x - \cos x}{\sin x + 2\cos x} \\ R_2(-\sin x, -\cos x) = \frac{-\sin x + \cos x}{-\sin x - 2\cos x} \\ R_1 = R_2 \Rightarrow \text{parna po sin i cos} \end{array} \right]$

$= \left[\begin{array}{l} \operatorname{tg} x = t \rightarrow \sin x = \frac{t}{\sqrt{1+t^2}} \\ dx = \frac{dt}{1+t^2} \rightarrow \cos x = \frac{1}{\sqrt{1+t^2}} \end{array} \right] =$

$$= \int \frac{\frac{t}{\sqrt{1+t^2}} - \frac{1}{\sqrt{1+t^2}}}{\frac{t}{\sqrt{1+t^2}} + 2\frac{1}{\sqrt{1+t^2}}} \cdot \frac{dt}{1+t^2} = \int \frac{\frac{t-1}{\cancel{\sqrt{1+t^2}}}}{\frac{t+2}{\cancel{\sqrt{1+t^2}}}} \cdot \frac{dt}{1+t^2} =$$

metod neodređenih koeficijenata

$$= \int \frac{t-1}{(t+2)(1+t^2)}\,dt = \left[\begin{array}{l} A = -\frac{3}{5} \\ B = \frac{3}{5} \\ C = -\frac{1}{5} \end{array} \right] = -\frac{3}{5}\int \frac{dt}{t+2} + \frac{1}{5}\int \frac{3t-1}{1+t^2}\,dt =$$

$$= -\frac{3}{5}\ln|t+2| + \frac{3}{5}\cdot\frac{1}{2}\ln|z| - \frac{1}{5}\operatorname{arctg} t + C$$

V $\int \sin^m x \cdot \cos^n x \, dx$, m - neparno
n - parno ili nula

Smjena: $\cos x = t$

(63) $\int \sin^3 x \cdot \cos^4 x \, dx = \int \sin^2 x \cdot \sin x \cdot \cos^4 x \, dx =$

$= \left[\begin{array}{l} \cos x = t \\ \sin x \, dx = -dt \\ \sin^2 x = 1 - \cos^2 x \end{array} \right] = \int (1-\cos^2 x) \cos^4 x \, \sin x \, dx =$

$= -\int (1-t^2) t^4 \, dt = -\int t^4 dt + \int t^6 dt =$

$= -\frac{t^5}{5} + \frac{t^7}{7} + C = -\frac{\sin^5 x}{5} + \frac{\sin^7 x}{7} + C$

(*) $\int \sin^5 x \, dx \rightarrow$ isto smjena $\cos x = t$

VI $\int \sin^m x \cdot \cos^n x \, dx$, m - parno ili nula
n - neparno

Smjena: $\sin x = t$

(*) $\int \sin^2 x \cos^5 x \, dx$

(*) $\int \cos^3 x \, dx$

~~VII $\int \sin^m x$~~

VII $\int \sin^m x \cdot \cos^n x \, dx$, m i n → parni

$\Rightarrow \sin^2 x = \frac{1-\cos 2x}{2}$

$\Rightarrow \cos^2 x = \frac{1+\cos 2x}{2}$

(64) $\int \sin^2 x \cdot \cos^4 x \, dx = \int \frac{1-\cos 2x}{2} \cdot \left(\frac{1+\cos 2x}{2}\right)^2 dx =$

$= \frac{1}{8} \int (1-\cos^2 2x)(1+\cos 2x) \, dx =$

$= \frac{1}{8}\int (1+\cos 2x - \cos^2 2x - \cos^3 2x)\,dx =$

$= \frac{1}{8}\int dx + \frac{1}{8}\int \cos 2x\,dx - \frac{1}{8}\int \cos^2 2x\,dx - \frac{1}{8}\int \cos^3 2x\,dx =$

$= \left|\begin{array}{l} \text{smjena} \to \sin 2x = t \\ 2\cos 2x\,dx = dt \\ \cos 2x\,dx = \frac{1}{2}dt \end{array}\right| =$

$= \frac{1}{8}x + \frac{1}{16}\sin 2x - \frac{1}{8}\int \frac{1+\cos 4x}{2}\,dx - \frac{1}{16}\int (1-t^2)\,dt =$

$= \ldots \frac{1}{16}x - \frac{1}{64}\sin 4x + \frac{1}{48}\sin^3 2x + C$

VIII) $\int \frac{\sin^m x}{\cos^n x}\,dx$ ili $\int \frac{\cos^m x}{\sin^n x}\,dx$, m, n parno

smjena $\operatorname{tg} x = t$

(65) $\int \frac{\sin^2 x}{\cos^6 x}\,dx = \left[\begin{array}{ll} \operatorname{tg} x = t & \sin x = \frac{t}{\sqrt{1+t^2}} \\ dx = \frac{dt}{1+t^2} & \cos x = \frac{1}{\sqrt{1+t^2}} \end{array}\right] =$

< . . .

(66) $I_n = \int \frac{dx}{(x^2+a^2)^n} = \left[\begin{array}{l} U = \frac{1}{(x^2+a^2)^n} \\ dU = -n\,(x^2+a^2)^{-n-1}\cdot 2x\,dx \\ dU = \frac{-2n\,x}{(x^2+a^2)^{n+1}}\,dx \\ V = \int dx = x \end{array}\right] =$

$= \frac{x}{(x^2+a^2)^n} + 2n\int \frac{x^2}{(x^2+a^2)^{n+1}}\,dx =$

Rekurentna formula

$$= \frac{x}{(x^2+a^2)^n} + 2n\int \frac{x^2+a^2-a^2}{(x^2+a^2)^{n+1}}\,dx =$$

$$= \frac{x}{(x^2+a^2)^n} + 2n\int \frac{dx}{(x^2+a^2)^n} - 2na^2\int \frac{dx}{(x^2+a^2)^{n+1}}$$

$$I_n = \frac{x}{(x^2+a^2)^n} + 2n\,I_n - 2na^2\,I_{n+1}$$

$$I_{n+1} = \frac{1}{2na^2}\left(\frac{x}{(x^2+a^2)^n} + (2n-1)I_n\right), \quad n \geq 1$$

$$I_{n+1} = \frac{x}{2na^2(x^2+a^2)^n} + \frac{2n-1}{2na^2}\,I_n, \quad n \geq 1$$

potrebno znati I_1

$$I_1 = \int \frac{dx}{x^2+a^2} = \left[\begin{matrix} x = at \\ dx = a\,dt \end{matrix}\right] = \int \frac{a\,dt}{a^2t^2+a^2} = \frac{1}{a}\,\mathrm{arctg}\,\frac{x}{a}$$

---

(67) $I_n = \int \frac{dx}{\sin^n x} = \int \frac{\sin x}{\sin^{n+1} x}\,dx =$

$$= \left[\begin{matrix} U = \frac{1}{\sin^{n+1} x} \Rightarrow dU = -(n+1)\left(\sin^{-(n+2)} x\right)\cdot \cos x\,dx \\ dU = -\frac{(n+1)\cos x}{\sin^{n+1} x}\,dx \\ V = \int \sin x\,dx \Rightarrow -\cos x \end{matrix}\right] =$$

$$= \frac{-\cos x}{\sin^{n+1} x} - (n+1)\int \frac{\overset{1-\sin^2 x}{\cos^2 x}}{\sin^{n+1} x}\,dx =$$

$$= \frac{-\cos x}{\sin^{n+1} x} - (n+1)\int \frac{dx}{\sin^{n+2} x} + (n+1)\int \frac{dx}{\sin^n x}$$

$$I_n = -\frac{\cos x}{\sin^{n+1} x} - (n+1)\,I_{n+2} + (n+1)\,I_n$$

$$(n+1) I_{n+2} = \frac{-\cos x}{\sin^{n+1} x} + n I_n$$

$$I_{n+2} = \frac{-\cos x}{\sin^{n+1} x\,(n+1)} + \frac{n}{n+1} I_n, \quad n \geq 0$$

$$I_n = \frac{-\cos x}{\sin^{n+1} x \cdot (n+1)} + \frac{n}{n+1} I_n, \quad n \geq 0$$

$$I_n = \frac{-\cos x}{(n+1)\sin^{n-1} x} + \frac{n-2}{n+1} I_{n-2}, \quad n \geq 2$$

$$I_0 = \int dx = x, \quad I_1 = \int \frac{dx}{\sin x} = \left[\begin{array}{l} \operatorname{tg}\frac{x}{2} = t \\ dx = \frac{2\,dt}{1+t^2} \\ \sin x = \frac{2t}{1+t^2} \end{array}\right] =$$

$$= \int \frac{\frac{2\,dt}{1+t^2}}{\frac{2t}{1+t^2}} = \int \frac{dt}{t} = \ln|t| + C = \ln\left|\operatorname{tg}\frac{x}{2}\right| + C$$

Razni integrali

(68) $$\int \frac{dx}{x^4+1} = \int \frac{dx}{x^4+2x^2+1-2x^2} = \int \frac{dx}{(x^2+1)^2-2x^2} =$$

$$= \int \frac{dx}{(x^2+1-\sqrt{2}x)(x^2+1+\sqrt{2}x)} = \text{metod neodređenih koef.}$$

(69) $$I = \int \frac{x}{(x^2-3x+2)\sqrt{x^2-4x+5}}\,dx \qquad \left(\int \frac{Mx+N}{(x-\alpha)^k \cdot \sqrt{\ }}\right)$$

$$P = \frac{x}{x^2-3x+2} = \frac{x}{(x-1)(x-2)} = \frac{2}{x-2} - \frac{1}{x-1}$$

neodređ. koef.

$$I = \int \left(\frac{2}{x-2} - \frac{1}{x-1} \right) \cdot \frac{1}{\sqrt{x^2-4x+5}}\, dx =$$

$$= 2\int \frac{dx}{(x-2)\sqrt{x^2-4x+5}} - \int \frac{dx}{(x-1)\sqrt{x^2-4x+5}}$$

$x-2 = \frac{1}{t}$ $\qquad\qquad$ $x-1 = \frac{1}{z}$ ...

---

(70) $$\int \frac{dx}{\sqrt[4]{(x-1)^3(x+2)^5}} = \left[\sqrt[4]{(x-1)^4 \cdot (x+2)^4 \cdot \frac{x+2}{x-1}} \right] =$$

$$= (x-1)(x+2) \cdot \sqrt[4]{\frac{x+2}{x-1}} \longrightarrow \int R\left(x, \left(\frac{ax+b}{cx+d}\right)^{\frac{m}{n}}, \right)$$

$$\left(\frac{x+2}{x-1}\right)^{\frac{\cancel{1}}{4}} = t^4 \qquad \frac{ax+b}{cx+d} = t^k, \; k \text{ najmanji zajed. sadrž.}$$

~~C + ...ac~~

$$I = \int \frac{dx}{(x-1)(x+2)\sqrt[4]{\frac{x+2}{x-1}}}$$

---

(71) $$\int \frac{1 + x\cos x}{\sin^3 x}\, dx = \int \frac{dx}{\sin^3 x} + \int \frac{x\cos x}{\sin^3 x}\, dx =$$

$$= \int \frac{\sin x}{\sin^4 x}\, dx + \int \frac{x \cos x}{\sin^3 x}\, dx$$

$$\left[\begin{array}{l} \cos x = t \\ \sin x\, dx = -dt \end{array} \right] \qquad \left[\begin{array}{l} U = x \Rightarrow dU = dx \\ V = \int \frac{\cos x}{\sin^3 x}\, dx = -\frac{1}{2} \frac{1}{\sin^2 x} \end{array} \right] =$$

$$= -\int \frac{dt}{(1-t)^2(1+t^2)} - \frac{x}{2\sin^2 x} - \frac{1}{2}\operatorname{ctg} x$$

$\longrightarrow$ metod neodređenih koeficijenata

(12) $\int \ln\left(\frac{x^{\sqrt{3x}}}{\sqrt{x+2}+\sqrt{x}}\right)dx = \int \ln x^{\sqrt{3x}}\,dx - \int \ln(\sqrt{x+2}+\sqrt{x})\,dx =$

$= \int \sqrt{3x}\ln x\,dx - \int \ln(\sqrt{x+2}+\sqrt{x})\,dx =$

$\left[\begin{array}{l} U=\ln x \Rightarrow dU = \frac{1}{x}dx \\ V=\int\sqrt{x}\,dx = \frac{2}{3}\sqrt{x^3} \end{array}\right]$ $\left[\begin{array}{l} U=\ln(\sqrt{x+2}+\sqrt{x}) \\ dU = \frac{1}{\sqrt{x+2}+\sqrt{x}}\cdot\left(\frac{1}{2\sqrt{x+2}}+\frac{1}{2\sqrt{x}}\right)dx \\ V=x \quad \rightarrow dU = \frac{dx}{\sqrt{x}\cdot 2\sqrt{x+2}} \end{array}\right]$

$= \sqrt{3}\left(\frac{2}{3}\sqrt{x^3}\ln x - \frac{2}{3}\int\sqrt{x}\,dx\right) - \left(x\ln(\sqrt{x+2}+\sqrt{x}) - \frac{1}{2}\int\frac{x}{\sqrt{x+2}\sqrt{x}}\,dx\right) =$

$= \frac{2\sqrt{x}}{3}\sqrt{x^3}\ln x - \frac{2\sqrt{3}}{3}\cdot\frac{2}{3}\sqrt{x^3} - x\ln(\sqrt{x+2}+\sqrt{x}) - \frac{1}{2}\int\frac{x\,dx}{\sqrt{x^2+2x}}$

$\int \frac{Ax+B}{\sqrt{\ }}$

---

(13) $\int \frac{\sqrt{\mathrm{ctg}\,x}}{1+\sin 2x - 4\cos^2 x}\,dx = \left[\begin{array}{ll} \mathrm{tg}\,x = t & \sin x = \frac{t}{\sqrt{1+t^2}} \\ dx = \frac{dt}{1+t^2} & \cos x = \frac{1}{\sqrt{1+t^2}} \end{array}\right] =$

$= \int \frac{\frac{1}{\sqrt{t}}}{1+\underbrace{2\frac{t}{\sqrt{1+t^2}}\cdot\frac{1}{\sqrt{1+t^2}}}_{2\sin x\,\cdot\,\cos x} - \frac{1}{1+t^2}}\cdot\frac{dt}{1+t^2} =$

$= \int \frac{\frac{1}{\sqrt{t}}}{\frac{1+t^2+2t-4}{1+t^2}}\cdot\frac{dt}{1+t^2} = \int \frac{dt}{\sqrt{t}\,(t^2+2t-3)} =$

$$= \left[\sqrt{t} = z;\ \frac{1}{2\sqrt{t}}\,dt = dz;\ \frac{dt}{\sqrt{t}} = 2\,dz \right] =$$

$$= 2\int \frac{dz}{z^4+2z^2-3} = 2\cdot\int \frac{dz}{(z^2-1)(z^2+3)} =$$

$$= 2\int \frac{dz}{(z-1)(z+1)(z^2+3)}$$ → metod neodređenih koeficijenata

---

(74) $$\int \frac{\sin 2x \cdot \sqrt[3]{\cos x+2}}{\sqrt[3]{\cos x+2}+\cos x}\,dx = \left[\begin{array}{l} \cos x = t \\ \sin x\,dx = -dt \end{array} \right] =$$

$$= -2\int \frac{t\,\sqrt[3]{t+2}}{\sqrt[3]{t+2}+t}\,dt = \left[\begin{array}{l} R\left(t,(t+2)^{\frac{1}{3}}\right) \\ t+2 = z^3 \\ dt = 3z^2\,dz \\ t = z^3-2 \end{array} \right] =$$

$$= -2\int \frac{(z^3-2)\cdot z}{z+z^3-2}\,3z^2\,dz =$$

$$= -6\int \frac{z^6-2z^3}{z^3+z-2}\,dz$$ → podijelimo, smanjujemo, metoda neodređenih koeficijenata